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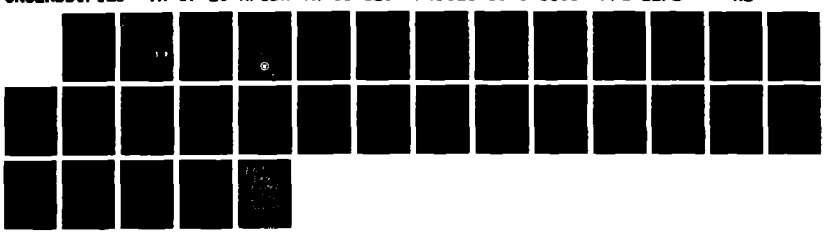

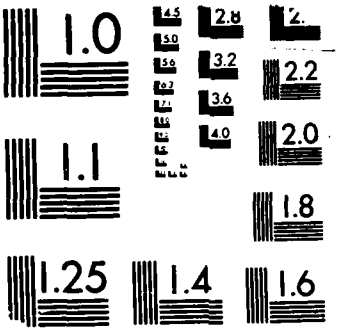

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20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

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FURTHER EXTENSIONS OF THE CHOQUET-DENY AND DENY THEOREMS WITH APPLICATIONS IN CHARACTERIZATION THEORY *

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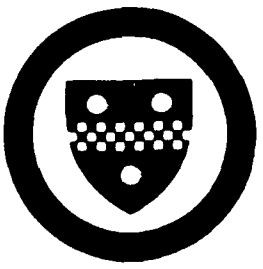

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ABSTRACT

Davies and Shanbhag (1987) identified, under some mild conditions. the solution to the convolution equation

$$H(x) = \int_S H(x+y)\mu(dy), \quad x \in S$$

where S is a Polish Abelian semigroup with zero element, H: $S \to \mathbb{R}_+$, a nonnegative continuous function and μ is a measure. A variant of the result in the case where H is bounded and μ is a certain bounded signed measure is obtained. This provides a generalized version of the Choquet-Deny theorem where μ is considered to be a probability measure and S to be a group.

The stability of the solution to the above equation is examined by adding an error term. Further, solutions to simultaneous equations of the above type are considered. The results of this paper generalize those obtained by Gu and Lau (1984), Lau and Rao (1982), Ramachandran *et al* (1987), Shimuzu (1978) and others for $S = \mathbb{R}$ or $\mathbb{R}_+$.

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1. INTRODUCTION

Let S be a Polish Abelian semigroup with zero element, $H: S \to \mathbf{R}_+$ a nonnegative continuous function and μ a measure on (the Borel σ-field of) S such that the following integral or (inverse) convolution equation is satisfied:

$$H(x) = \int_S H(x+y)\mu(dy), \quad x \in S. \tag{1.1}$$

Recently, Davies and Shanbhag (1987) studied (1.1) and showed in particular that if S is embeddable into a locally compact second countable Hausdorff Abelian group and certain mild conditions are satisfied, then the H in (1.1) has an integral representation as a weighted average of the μ-harmonic exponential functions on S. A function $e: S \to \mathbf{R}_+$ is called an exponential function if it is continuous and satisfies $e(x+y) = e(x)e(y)$ for all $x, y \in S$; an exponential function e is called μ-harmonic if it satisfies $\int_S e(x)\mu(dx) = 1$. A special case of this result with S itself as a locally compact second countable Hausdorff Abelian group was established earlier by Deny (1961) and its specialized version when H is bounded (with a modification that it is not necessarily nonnegative) and μ is a probability measure by Choquet and Deny (1960).

As shown by Shanbhag (1977), Shimizu (1978), Rao (1983), Lau and Rao (1982, 1984), Rao and Shanbhag (1986), Ramachandran (1982, 1984) and Davies and Shanbhag (1987), these results have applications not only in renewal theory but also in characterization theory of probability distributions. Several characterizations of the exponential, Weibull and geometric distributions (or their variants) based on record values, order statistics, hazard function and the mean residual life function

etc., follow from these results. The Rao-Rubin theorem (1964) on damage models in a global form leading to characterizations of several discrete probability distributions and related characterizations of point processes and characterizations of stable distributions based on a certain factorization can also be obtained from these results.

Shimizu (1978) and more recently Ramachandran *et al* (1987) have considered several modified versions of (1.1) in the case of $S = \mathbb{R}$ or $S = \mathbb{R}_+$ that have applications in characterization theory especially in connection with stable distributions. Shimizu (1980) and Gu and Lau (1984) have also discussed (1.1), in the specialized situation of $S = \mathbb{R}_+$, involving an additional error term; these latter results are useful in arriving at stability versions of certain characterization properties.

In the present paper we aim at extending the Davies-Shanbhag (1987) result to situations covering the problems considered by Shimizu (1978, 1980), Gu and Lau (1984) and Ramachandran *et al* (1987). In Section 2, we give the results when H is bounded and micron is replaced by a certain bounded signed measure subsuming the specialized versions of the results given earlier by Shimizu (1978) and Ramchandran *et al* (1987); these turn out to be also extended versions of the Choquet-Deny theorem. In Section 3, we consider an extension of the Davies-Shanbhag result for the case of two simultaneous equations yielding the results of Ramachandran *et al* (1987) and those arrived at by some of the authors as special cases. In Section 4, we obtain extended versions of the results involving error terms given earlier by Shimizu (1980) and Gu and Lau (1984), and finally in Section 5, we discuss some applications.

2. EXTENDED VERSIONS OF THE CHOQUET-DENY THEOREM

The following results are various extensions of the Choquet-Deny theorem:

THEOREM 1. Let S be a Polish Abelian semigroup with zero element, $H: S \to \mathbb{R}$ a bounded continuous function, and μ and ν subprobability measures on S such that $\mu + \nu$ is a probability measure on S and

$$H(x) = \int_S H(x+y)(\mu - \nu)(dy), \quad x \in S. \tag{2.1}$$

Also let

$$\sigma = \mu + \sum_{m=0}^{\infty} \nu^{*2} * \mu^{*m} \tag{2.2}$$

with the usual notation for convolution. (Clearly the measure σ defined here is a probability measure.) Then for each $x \in S$,

$$H(x) = \begin{cases} H(x+y) \text{ if } y \in S^*(\sigma) \text{ (and hence if } y \in \operatorname{supp}(\mu)), \\ -H(x+y) \text{ if } y \in \operatorname{supp}(\nu), \end{cases} \tag{2.3}$$

where the notation $\operatorname{supp}(\cdot)$, as usual, stands for the support of the measure (concerned) and the notation $S^*(\cdot)$ stands for the smallest closed subsemigroup of S with zero element such that it includes the support of the measure.

Proof. We can rewrite (2.1) as

$$H(x) = \int_S H(x+y)\mu(dy) - \int_S H(x+y)\nu(dy), \quad x \in S \tag{2.4}$$

yielding in view of Fubini's theorem

$$H(x) = \int_S H(x+y)(\mu + \nu^{*2})(dy) - \int_S H(x+y)\,\nu * \mu(dy)$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$= \int_S H(x+y)\left(\mu + \sum_{m=0}^{n} \nu^{*2} * \mu^{*m}\right)(dy) - \int_S H(x+y)\nu * \mu^{*(n+1)}(dy),$$

$$n = 0,1,\ldots, \quad x \in S. \tag{2.5}$$

Observe that to arrive at (2.3), we have used successively the relation

$$\int_S H(x+y)\nu * \mu^{*n}(dy) = \int_S \left\{ \int_S H(x+y+z)\mu(dz) - \int_S H(x+y+z)\nu(dz) \right\} \nu * \mu^{*n}(dy)$$

$$= \int_S H(x+y)\nu * \mu^{*(n+1)}(dy) - \int_S H(x+y)\nu^{*2} * \mu^{*n}(dy),$$

$$n = 0,1,\ldots, \quad x \in S.$$

Using, in particular, the fact that H is bounded and $\mu(S) + \nu(S) = 1$, we see that for each $x \in S$, the expression on the right-hand side of (2.5) tends as $n \to \infty$ to $\int_S H(x+y)\sigma(dy)$ with σ as the probability measure given by (2.2). We have then the integral equation

$$H(x) = \int_S H(x+y)\sigma(dy), \quad x \in S. \tag{2.6}$$

The proof of Corollary 1 of Theorem 1 on p.21 of Davies and Shanbhag (1987), except for its last sentence, remains valid in the case of the equation (2.6) and yields that

$$H(x+y) = H(x), \quad x \in S, \quad y \in S^*(\sigma). \tag{2.7}$$

The equation (2.7) establishes the first part of the assertion (2.3) and in turn implies

$$H(x+y) = H(x), \quad x \in S, \quad y \in \mathrm{supp}(\mu) \cup \mathrm{supp}(\nu^{*2}) \tag{2.8}$$

giving in view of (2.1)

$$H(x)\nu(S) = -\int_S H(x+y)\nu(dy), \quad x \in S. \tag{2.9}$$

We have now to establish only the second part of the assertion (2.3). This result is trivially valid for $\nu(S) = 0$. We can therefore assume that $\nu(S) > 0$. If $z \in \mathrm{supp}(\nu)$, then $z + \mathrm{supp}(\nu) \subset \mathrm{supp}(\nu^{*2})$ and hence we get from (2.8) and (2.9)

$$\begin{aligned} H(x+z)\nu(S) &= -\int_S H(x+z+y)\nu(dy) \\ &= -H(x)\nu(S), \quad x \in S, \end{aligned}$$

which implies in view of the assumption $\nu(S) > 0$ that

$$H(x+z) = -H(x), \quad x \in S.$$

Consequently, we have the second part of the assertion in question, and hence the theorem. □

COROLLARY 1. The assertion of the theorem holds if the assumption that H is bounded is replaced by the weaker assumption that $H(\cdot + y) - H(\cdot)$ is bounded for each fixed $y \in \mathrm{supp}(\mu) \cup \mathrm{supp}(\nu)$ (with the implicit requirement that H is $(\mu+\nu)$-integrable) provided $\nu(S) > 0$.

Proof. For every $y \in \mathrm{supp}(\mu) \cup \mathrm{supp}(\nu)$, on applying the theorem to $H(\cdot + y) - H(\cdot)$, we have

$$H(\cdot+y+z) - H(\cdot+z) = \begin{cases} H(\cdot+y) - H(\cdot) & \text{if } z \in S^*(\sigma) \\ -\big(H(\cdot+y) - H(\cdot)\big) & \text{if } z \in \text{supp}(\nu). \end{cases} \tag{2.10}$$

From the first equation in (2.10), we get for every $x \in S$, $y \in \text{supp}(\nu)$ and $z \in S^*(\sigma)$

$$H(x+y+z) - H(x+y) = H(x+z) - H(x)$$

and from the second equation in (2.10), we get for every $x \in S$, $y \in \text{supp}(\nu)$ and $z \in \text{supp}(\mu)$

$$H(x+y+z) - H(x+y) = -\big(H(x+z) - H(x)\big).$$

Consequently, it follows that $H(x+z) - H(x) = 0$ for every $z \in \text{supp}(\mu)$ and $x \in S$. On integrating the second equation in (2.10) over y with respect to measure ν and simplifying, we get in view of (2.1)

$$\int_S \big(H(\cdot+y+z) + H(\cdot+y)\big)\mu(dy) - \big(H(\cdot+z) + H(\cdot)\big)$$

$$= \nu(S)\{H(\cdot+z) + H(\cdot)\}, \quad z \in \text{supp}(\nu). \tag{2.11}$$

Since $H(\cdot+y) = H(\cdot)$, $y \in \text{supp}(\mu)$, (2.11) implies readily that

$$-\nu(S)\{H(\cdot+z) + H(\cdot)\} = \nu(S)\{H(\cdot+z) + H(\cdot)\}, \quad z \in \text{supp}(\nu),$$

which implies in view of the assumption $\nu(S) > 0$, the second part of the assertion (2.1). The first part of the assertion is now obvious in view of the continuity of H and what is already arrived at. □

COROLLARY 2. Let S, μ, ν, H, σ and $S^*(\sigma)$ be as in Corollary 1. If there exists a dense subset A of S such that for every $x \in A$ we have a $y \in S^*(\sigma)$ with $x + y \in S^*(\sigma)$, then

$$H(x) = H(0), \quad x \in S$$

and either $H(0) = 0$ or $\nu(S) = 0$.

Proof. From the theorem we get

$$H(x+y) = H(x), \qquad x \in S, \qquad y \in S^*(\sigma). \tag{2.12}$$

This implies that $H(x) = H(x+y) = H(0)$ for every $x \in A$ and $y \in S^*(\sigma)$ and hence in view of the continuity of H, we have $H(x) = H(0)$ for all $x \in S$. From the equation (2.1) or from the second assertion of the theorem, it is then clear that either $H(0) = 0$ or $\nu(S) = 0$. □

Remark 1. Corollary 2 yields the Choquet-Deny theorem as a special case and hence this result, as well as Theorem 1 and Corollary 1, could be considered as extensions of the theorem in question. Theorem 1 and Corollary 1, for special cases of $S = \mathbb{R}_+$ and $S = \mathbb{R}$, have been dealt with either by Shimizu (1978) or by Ramachandran *et al* (1987); the arguments used to establish the results for the generalized situation of the present paper have been adapted from Ramachandran *et al* (1987) with the required modifications.

3. TWO SIMULTANEOUS INTEGRAL EQUATIONS

Let S be a Polish Abelian semigroup with zero element, $G: S \to \mathbb{R}_+$ and $H: S \to \mathbb{R}_+$ continuous functions, and μ and ν measures on S such that

$$\left.\begin{aligned} G(x) &= \int_S G(x+y)\mu(dy) + \int_S H(x+y)\nu(dy), \quad x \in S \\ H(x) &= \int_S H(x+y)\mu(dy) + \int_S G(x+y)\nu(dy), \quad x \in S \end{aligned}\right\}. \tag{3.1}$$

The equations (3.1) for $S = \mathbb{R}_+$ and $S = \mathbb{R}$ have been dealt with by many authors including Ramachandran *et al* (1987) in connection with the characterization problems of probability distributions; we shall provide the necessary details in Section 5 of the paper. It is interesting to note that (3.1) with $\nu(S) = 0$ reduces to a set of two equations of the type considered by Davies and Shanbhag (1987) and hence (3.1) may be viewed as a generalization of the Davies-Shanbhag integral equation. From what is shown in Davies and Shanbhag (1987), it is clear that to have the integral representations of Deny or something close to it for G and H in (3.1), we require some further conditions to be met.

Following the notation of the last section, we shall now introduce the condition given below:

CONDITION A. Let $\hat{S}$ be the smallest closed subsemigroup of S with zero element such that it includes $\text{supp}(\mu) \cup \big(\text{supp}(\nu) + \text{supp}(\nu)\big)$. The $\hat{S}$ satisfies $\text{supp}(\nu) \subset \hat{S}$ and there exist subsets B and D of S with the following properties:

(a) $B \in \mathcal{B}(S), \quad (\mu+\nu)(S \setminus B) = 0,$

(b) the subsemigroup generated by $D \cup S^*(\mu+\nu)$ is dense in S,

(c) for every $x \in D$ and $z \in B \setminus \{0\}$, we have $n \geq 0$, $k \geq 1$ and $y_1, \ldots, y_k$

in $S^*(\mu+\nu)$ such that

$$x + nz + y_1 + \ldots + y_{r-1} \in S + y_r, \quad r = 1,2,\ldots,k$$

and

$$x + nz + y_1 + \ldots + y_k \in S^*(\mu+\nu).$$

We have then the following theorem.

THEOREM 2. Suppose $\mu(\{0\}) + \nu(\{0\}) < 1$, and the Condition A is met. Then either $H(x) = G(x) = 0$ for all $x \in S$ or we have a probability space $(\Omega, \mathcal{Q}, P)$ and product measurable functions $\xi_i : S\times\Omega \to \mathbb{R}_+$, $i = 1,2$ such that $G(x) = G(0)E\big(\xi_1(x,\cdot)\big)$, $H(x) = H(0)E\{\xi_2(x,\cdot)\}$, $x \in S$ and the following conditions are met:

(i) $\xi_i(x+y,\cdot) = \xi_i(x,\cdot)\xi_i(y,\cdot)$ a.s. [P] for all $x,y \in S^*(\mu+\nu)$, $i = 1,2$.

(ii) $\int_S \xi_i(x,\cdot)(\mu+\nu)(dx) = 1$ a.s. [P], $i = 1,2$.

(iii) $\xi_1(x,\cdot) \equiv 0$ if $G(x) = 0$ and $G(0) \neq 0$ and $\xi_2(x,\cdot) \equiv 0$ if $H(x) = 0$ and $H(0) \neq 0$.

Proof. Observe that (3.1) implies

$$G(x) + H(x) = \int_S \big(G(x+y) + H(x+y)\big)(\mu+\nu)(dy), \quad x \in S. \tag{3.2}$$

Consequently from Lemma 5 of Davies and Shanbhag (1987), it follows that $G(x) + H(x) = 0$ for all $x \in S$ if $G(0) + H(0) = 0$. We now need only the case of $G(0) + H(0) > 0$ and hence without loss of generality that of $G(0) + H(0) = 1$ to be considered. From the argument leading to (41) in the proof of Theorem 2 of Davies and Shanbhag (1987), it follows that the present result is valid in the case of $\nu(S) = 0$. Now, as in Davies and Shanbhag (1987), we can assume without loss of generality the measure

$\mu + \nu$ in (3.2) to be σ-finite and hence both measures μ and ν to be σ-finite. Using Fubini's theorem and an argument essentially of the type used in establishing (2.5), we get in view of (3.1) that

$$G(x) = \int_S G(x+y)(\mu + \sum_{m=0}^{n} \nu^{*2} * \mu^{*m})(dy)$$

$$+ \int_S H(x+y)\nu * \mu^{*(n+1)}(dy), \quad x \in S. \tag{3.3}$$

(It may be noted here that by $\nu^{*2} * \mu^{*m}$, we really mean the measure on S given by

$$\int_S \cdots \int_S I_{\{\underset{\sim}{x}(=(x_1,\ldots,x_{m+2})) : x_1+\ldots+x_{m+2} \in B\}} \nu(dx_1)\nu(dx_2)\mu(dx_3)\ldots\mu(dx_{m+2})$$

for each Borel set B; the measure $\nu * \mu^{*(n+1)}$ is defined analogously.)

We have in view of (3.2),

$$\int_S H(x+y)\mu * \nu^{*(n+1)}(dy) \leq \int_S \Big(G(x+y) + H(x+y)\Big)\nu * \mu^{*(n+1)}(dy)$$

$$= \int_S E^*\Big(\xi(x+y,\cdot)\Big)\nu * \mu^{*(n+1)}(dy),$$

$$x \in S, \quad n \geq 0, \tag{3.4}$$

where ξ is the ξ_1 of our theorem when $G + H$ is taken in place of G, $\mu + \nu$ is taken in place of μ, and $\nu \equiv 0$, and E^* is the corresponding expectation relative to the concerned probability measure. The right-hand side of (3.4) equals for each $x \in S$ and $n \geq 0$

$$E^*\Big(\xi(x,\cdot)\Big(\int_S \xi(y,\cdot)\mu(dy)\Big)^{n+1}\int_S \xi(y,\cdot)\nu(dy)\Big)$$

with

$$0 \leq \int_S \xi(y,\cdot)\mu(dy) = 1 - \int_S \xi(y,\cdot)\nu(dy) \leq 1 \quad \text{a.s. } [P_H^*]$$

(in obvious notation) and

$$E^*\big(\xi(x,\cdot)\big)\big(= G(x) + H(x)\big) < \infty \quad \text{for all} \quad x \in S.$$

The Lebesgue dominated convergence theorem implies then that the right-hand side of (3.4) and hence the second integral in (3.3) tends to zero as $n \to \infty$. We then get from (3.3)

$$G(x) = \int_S G(x+y)(\mu + \sum_{m=0}^{\infty} \nu^{*2} * \mu^{*m})(dy), \quad x \in S. \tag{3.5}$$

(Note that to obtain (3.5) from (3.1), the part $\text{supp}(\nu) \subset \hat{S}$ in Condition A is not used.) The same equation is also valid for H by symmetry. Suppose we denote the measure appearing in (3.5) by σ. In view of (3.5) and Theorem 1 of Davies and Shanbhag (1987), we have either $G(0) = 0$ or there exists a probability space $(\Omega_1, \mathcal{A}_1, P^{(1)})$ and a product measurable function $\xi^{(1)}: S \times \Omega_1 \to \mathbb{R}_+$ such that all the conditions appearing in Theorem 1 of Davies and Shanbhag (1987) (with obvious notational changes) are satisfied; an analogous result is also valid for H. From the equations corresponding to (17), (18) and (19) in Theorem 1 of Davies and Shanbhag (1987), and the stipulation that $\text{supp}(\nu) \subset \hat{S}$, we get in view of Fubini's theorem

$$\Big(1 - \int_S \xi^{(i)}(y,\cdot)\mu(dy)\Big)^2 = \Big(\int_S \xi^{(i)}(y,\cdot)\nu(dy)\Big)^2 \quad \text{a.s. } [P^{(i)}],$$

$$i = 1,2. \tag{3.6}$$

(In view of (19), Fubini's theorem works here; for the details as to how the theorem applies, see Section 2 and especially Lemma 1 of Davies and Shanbhag (1987).) From (3.6), we get

$$\int_S \xi^{(i)}(y,\cdot)(\mu+\nu)(dy) = 1 \quad \text{a.s. } [P^{(i)}], \quad i = 1,2. \tag{3.7}$$

On taking the expectations in (3.7) and using Fubini's theorem once again, we get in view of (16) of Davies and Shanbhag (1987)

$$\left.\begin{aligned} \int_S G(y)(\mu+\nu)(dy) &= G(0) \\ \int_S H(y)(\mu+\nu)(dy) &= H(0) \end{aligned}\right\}. \tag{3.8}$$

If $G(x) > 0$, then applying the first result of (3.8) to $G(x+\cdot)$, we see that

$$\int_S G(x+y)(\mu+\nu)(dy) = G(x). \tag{3.9}$$

This last identity is also valid if $G(x) = 0$ since in that case, in view of Condition A, it follows trivially from (3.5) that $G(x+y) = 0$ for every $y \in \text{supp}(\mu+\nu)$ (indeed from Lemma 5 of Davies and Shanbhag (1987) it follows that this is so for every $y \in S$). Thus, it follows that (3.9) is valid for every $x \in S$. The identity is also valid for H by symmetry. Since the version of the present theorem is valid for the special case of $\nu(S) = 0$, we have now probability spaces $(\Omega_i, \mathcal{A}_i, P_i)$, $i = 1,2$ and product measurable functions $\xi_i^*: S\times\Omega_i \to \mathbb{R}_+$, $i = 1,2$ such that the analogues of (i) and (ii) with P replaced by P_i and the obvious result corresponding to (iii) hold. Considering $(\Omega, \mathcal{A}, P)$ to be the product probability space corresponding to $(\Omega_i, \mathcal{A}_i, P_i)$, $i = 1,2$ and defining

$$\xi_i\big(x,(\omega_1,\omega_2)\big) = \xi_i^*(x,\omega_i), \quad i = 1,2, \quad x \in S, \quad \omega_1 \in \Omega_1, \quad \omega_2 \in \Omega_2,$$

we arrive at the required result.

Remark 2. In view of the remark made immediately after (3.5) and the crucial role played by (3.5) in the proof of Theorem 2, it should be possible to give other versions of Theorem 2 altering the requirements in Condition A.

Remark 3. If S is countable, clearly Theorem 2 is valid with a.s. [P] deleted in (ii) and (iii). This result yields among other things a recent result of Ressel (1985) as a special case.

Remark 4. Suppose S is a closed subsemigroup of a locally compact second countable Hausdorff Abelian group and we have a compact subset K of S, which is the closure of a nonempty open subset of the group, and a subset D^* dense in $\text{supp}(\nu+\mu)$ with the property that for every $x \in D^* \setminus \{0\}$ there exists an $x' \in K$ such that $x \in S + x'$. Then Theorem 2 remains valid with a.s. $[P_i]$ in (i) and (ii) deleted and also $\xi_i(\cdot,\omega)$ as $(\mu+\nu)$-harmonic exponential functions on S for each $\omega \in \Omega$. This follows from the argument of Davies and Shanbhag (1987) appearing on pages 27 and 28 in the proof of Theorem 2.

COROLLARY 3. Let $n \geq 1$, $S = \Pi_{i=1}^{n} S_i$ with $S_i = \mathbb{Z}(= \{0, \pm 1, \pm 2, \ldots\})$ or $\mathbb{N}_0(= \{0,1,2,\ldots\})$ or $-\mathbb{N}_0$ or $\mathbb{R}$ or $\mathbb{R}_+$ or $-\mathbb{R}_+$ and let λ be the restriction to S of a Haar measure on the smallest subgroup of $\mathbb{R}^n$ containing S. Let $g: S \to \mathbb{R}_+$ and $h: S \to \mathbb{R}_+$ be Borel measurable functions that are locally integrable with respect to λ, and μ and ν be a σ-finite measure on S with $\mu(\{0\}) + \nu(\{0\}) < 1$. Assume that Condition A is met. Then

$$g(\cdot) = \int_S g(\cdot + y)\mu(dy) + \int_S h(\cdot + y)\nu(dy) \quad \text{a.e. } [\lambda]$$

$$h(\cdot) = \int_S h(\cdot + y)\mu(dy) + \int_S g(\cdot + y)\nu(dy) \quad \text{a.e. } [\lambda]$$

implies that

$$g(\cdot) = \int_{[-\infty,\infty]^n} e^{\langle \cdot, x \rangle} \alpha_1(dx) \quad \text{a.e. } [\lambda]$$

and

$$h(\cdot) = \int_{[-\infty,\infty]^n} e^{\langle \cdot, x \rangle} \alpha_2(dx) \quad \text{a.e. } [\lambda]$$

where α_1 and α_2 are measures on $[-\infty,\infty]^n$ such that

$$\alpha_i(\{x \in [-\infty,\infty]^n : \int_S e^{\langle x,y \rangle}(\mu + \nu)(dy) \neq 1 \text{ or}$$

$$\langle x,y \rangle \text{ is undefined for some } y \in S\}) = 0, \quad i = 1,2$$

(and we define $e^{-\infty} = 0$, $e^{\infty} = \infty$ and $0 \cdot \infty = 0$).

Proof. The result follows essentially from the arguments of Davies and Shanbhag (1987) used to prove Corollary 2 of Theorem 2. Alternatively one could prove this result as follows. Without loss of generality, we can assume $S = \Pi_{i=1}^n S_i$ with the first m of the S_i's to be either equal to $\mathbb{Z}$ or $\mathbb{N}_0$ and the remaining S_i's to be equal to $\mathbb{R}$ or $\mathbb{R}_+$ where m is a fixed integer such that $0 \leq m \leq n$. For every positive integer k, take a_k to be the point in $\mathbb{R}_+^n$ that has first m coordinates to be equal to zero and the remaining coordinates equal to k^{-1}. Observe that for each k, $G_k(\cdot) = \{\int_{Q_k} g(\cdot + y)\lambda(dy)\}/\lambda(Q_k)$ and $H_k(\cdot) = \{\int_{Q_k} h(\cdot + y)\lambda(dy)\}/\lambda(Q_k)$ with $Q_k = [0,a_k]$ are continuous functions satisfying the requirements of Theorem 2. In view of what is said in Remark 4, it follows that in the present case the result in terms of $(\mu + \nu)$-harmonic exponential functions is valid. Also in view of the local integrability of g and h, $G_k \to g$ and $H_k \to h$, a.e. $[\lambda]$. Using the relevant continuity theorem for Fourier transforms of measures, we can then conclude that the corollary is valid.

(This argument in the specialized situation of the Davies-Shanbhag problem was hinted in Remark 6 of Davies and Shanbhag (1987).)

Remark 5. There are some typographical/printing errors or minor inaccuracies in Davies and Shanbhag (1987). Although most of these are easily detectable, it may be noted in particular that in this paper in the last line of the proof of Lemma 1, B_0^c has appeared as B_0; in Remark 2, (3) has appeared in place of (10); in Remark 4, + is more appropriate than $\cup$ and Condition B is more appropriate than (d) of Condition A. Also the definition of N_4 appearing on p.28 should have been:

"The null set on which $\eta(x,\cdot) = \eta(0,\cdot)\xi(x,\cdot)$ for some x or $\xi(x+y,\cdot) \neq \xi(x,\cdot)\xi(y,\cdot)$ for all x, y in a fixed countable dense subset S' of S."

Remark 6. From Ramachandran *et al* (1987), it is evident that in the case of $S = \mathbb{R}$ or $\mathbb{R}_+$ or $\mathbb{Z}$ or $\mathbb{N}_0$ (or $-\mathbb{R}_+$ or $-\mathbb{N}_0$), except in the case of $\nu(S) = 0$, the G and H in Theorem 2 are equal. In general, however, this situation does not remain valid (although it is always true that $\int_S (H(\cdot + y)\nu(dy) = \int_S G(\cdot + y)\nu(dy)$). This is illustrated by the following example.

EXAMPLE. Let $S = \mathbb{N}_0^2$ and μ and ν be measurable on S such that μ has full support with its restriction to $\{(x,y): x \in \mathbb{N}_0, y = 0\}$ as a probability measure and μ has $\{(0,1)\}$ as its support. Define

$$H(x,y) = \begin{cases} c & \text{if} \quad x = 0,1,\ldots, \quad y = 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a fixed positive constant, and take $G \equiv 0$. It trivially follows that, in this case, the G and H satisfy (3.1) and also all the requirements of Theorem 2 are met. However, we do not have here G = H. The same point could obviously be illustrated from several other examples.

4. A STABILITY THEOREM

We now give the following stability theorem. The stability theorems are of importance in characterization theory of probability distributions since they are useful in assessing whether or not the distribution can be taken to be close to a certain distribution when it satisfies a characterization property of that distribution approximately. Since the integral equation (1.1) is involved in characterizations of several probability distributions, it should therefore be worthwhile to discuss the associated stability theorem:

THEOREM 3. Let S be an Abelian metric semigroup, $H: S \to \mathbb{R}_+$ a Borel measurable function $c \in [0,1)$ and μ a σ-finite measure on S such that

$$H(x) = \int_S H(x+y)\mu(dy) + \alpha(x), \quad x \in S, \tag{4.1}$$

where α is such that $|\alpha(x)| \leq \alpha^*(x)$ for all $x \in S$ with α^* as a real-valued Borel measurable function satisfying

$$\int_S \alpha^*(x+y)\mu(dy) \leq c\alpha^*(x), \quad x \in S. \tag{4.2}$$

Then the H can be expressed as

$$H(x) = H_1(x) + H_2(x), \quad x \in S, \tag{4.3}$$

where H_1 is a nonnegative Borel measurable function on S satisfying

$$H_1(x) = \int_S H_1(x+y)\mu(dy), \quad x \in S, \tag{4.4}$$

and H_2 is a Borel measurable function on S such that $|H_2(x)|$ is bounded by $\alpha^*(x)(1-c)^{-1}$ for each $x \in S$.

Proof. Using Fubini's theorem, we get successively for $n = 1,2,\ldots$

$$H(x) = \int_S H(x+y)\mu^{*n}(dy) + \sum_{m=0}^{n-1} \int_S \alpha(x+y)\mu^{*m}(dy), \quad x \in S. \tag{4.5}$$

(The requirement of the integrability of $\alpha(x+\cdot)$ with respect to the measure μ^{*m} for each $m = 0,1,\ldots$, and $x \in S$ is met in view of (4.2).) Also (4.2) implies that

$$\sum_{m=0}^{\infty} \int_S |\alpha(x+y)|\mu^{*m}(dy) \leq \alpha^*(x)(1-c)^{-1}, \quad x \in S. \tag{4.6}$$

In view of (4.6), it follows that

$$\sum_{m=0}^{n-1} \int_S \alpha^+(x+y)\mu^{*m}(dy)$$

and

$$\sum_{m=0}^{n-1} \int_S \alpha^-(x+y)\mu^{*m}(dy)$$

converge to finite limits as $n \to \infty$ for each $x \in S$ with obviously the limiting functions as Borel measurable functions. This observation, in turn, implies that

$$\sum_{m=0}^{n-1} \int_S \alpha(x+y)\mu^{*m}(dy)$$

converges as $n \to \infty$ to a Borel measurable function. Denote this function by H_2. Because of (4.6), it follows that $|H_2(x)| \leq \alpha^*(x)(1-c)^{-1}$ for each $x \in S$. In view of (4.5), it follows that $\int_S H(x+y)\mu^{*n}(dy)$ tends as $n \to \infty$ to $H(x) - H_2(x)$, a nonnegative Borel measurable function, for each $x \in S$. Denote this new function by H_1. From (4.5) and (4.6), it follows that

$$\int_S H(x+y)\mu^{*n}(dy) \leq H(x) + \alpha^*(x)(1-c)^{-1}, \quad x \in S, \quad n \geq 1.$$

Since

$$\int_S H(x+y)\mu^{*n}(dy) = \int_S \left\{ \int_S H(x+y+z)\mu^{*n-1}(dz) \right\} \mu(dy),$$

$$n = 1,2,\ldots, \quad x \in S$$

and $H(x+\cdot)$ and $\alpha(x+\cdot)$ are μ-integrable for each $x \in S$, the Lebesgue dominated convergence theorem implies that the H_1 satisfies (4.4). (The fact that the H_1 satisfies (4.4) could also be seen by noting that, in view of (4.1) and (4.6), we have for each $x \in S$, $H_1(x+\cdot)$ to be μ-integrable satisfying

$$\begin{aligned}\int_S H_1(x+1)\mu(dy) &= \int_S H(x+y)\mu(dy) - \int_S H_2(x+y)\mu(dy) \\ &= \int_S H(x+y)\ (dy) - \sum_{m=1}^{\infty} \int_S \alpha(x+y)\mu^{*m}(dy) \\ &= H(x) - \alpha(x) - \sum_{m=1}^{\infty} \int_S \alpha(x+y)\mu^{*m}(dy) \\ &= H_1(x).)\end{aligned}$$

COROLLARY 4. Let S and λ be as in Corollary 3 and $c \in [0,1)$. Let $h: S \to \mathbb{R}_+$ and $\alpha: S \to \mathbb{R}$ be Borel measurable functions that are locally integrable with respect to λ and μ be a σ-finite measure on S such that

$$h(\cdot) = \int_S h(\cdot + y)\mu(dy) + \alpha(\cdot) \quad \text{a.e. } [\lambda]$$

with $|\alpha(x)| \leq \alpha^*(x)$ for every $x \in S$ for some λ-locally integrable function α^* satisfying

$$\int_S \alpha^*(x+y)\mu(dy) \leq c\alpha^*(x) \quad \text{for a.e. } [\lambda], \quad x \in S.$$

Then the h can be represented as

$$h(\cdot) = h_1(\cdot) + h_2(\cdot) \quad \text{a.e. } [\lambda]$$

with

$$h_1(\cdot) = \int_{[-\infty,\infty]^n} e^{\langle \cdot, x \rangle} \nu(dx),$$

ν being a measure on $[-\infty,\infty]^n$ such that

$$\nu(\{x \in [-\infty,\infty]^n: \int_S e^{\langle x,y \rangle} \mu(dy) \neq 1 \text{ or } \langle x,y \rangle \text{ is undefined for some } y \in S\}) = 0,$$

and h_2 as a λ-locally integrable Borel measurable function on S satisfying $|h_2(x)| \leq \alpha^*(x)(1-c)^{-1}$ for every $x \in S$.

Proof. The result follows from the argument of the proof of Corollary 3. In this case, the H_k defined in the proof of Corollary 3 satisfies (4.1) with α replaced by α_k, where α_k is such that

$$\alpha_k(\cdot) = \{\int_{Q_k} \alpha(\cdot + y)\lambda(dy)\}/\lambda(Q_k), \quad k = 1,2,\ldots .$$

α_k obviously satisfies (4.2) in the statement of Theorem 3 with α_k^* analogously defined and we get in obvious notation

$$H_k(x) = H_{1k}(x) + H_{2k}(x), \quad x \in S, \quad k = 1,2,\ldots \tag{4.7}$$

with H_{1k} and H_{2k} satisfying the conditions corresponding to those stated in the statement of Theorem 3. The continuity theorem used in the proof of Corollary 3 implies, in view of the local integrability of h, α and α^*, that the present corollary is then valid. (Incidentally, in the statement of the corollary, the local integrability of α is implied by that of α^* and hence one could have avoided mentioning it specifically.)

Remark 7. In the case of $S = \mathbb{R}_+$, the result of Corollary 4 when μ is a probability measure and $\alpha^*(x) = c'e^{-\varepsilon x}$, $x \geq 0$ with $\varepsilon > 0$ is arrived at by Gu and Lau (1984) through their Theorem 1. However, it may be noted that the Gu-Lau paper contains some minor inaccuracies. For instance, in the proof of Theorem 1 as well as of Theorem 2 of the paper, in many places 'a.e.' is not mentioned even when the argument used required this to be there. It is also worth pointing out here that in Section 4 of the paper, Gu and Lau (1984) discuss three stability results concerning probability distributions through their Theorems 4.1, 4.2 and 4.3. Certain versions of these results follow as special cases of our Corollary 4. (Incidentally, we may point out here that since the error terms are usually small, ε^{-1} is better as a notation than the Gu-Lau ε mentioned above in the exponent of $\alpha^*(x)$.)

5. APPLICATIONS OF RESULTS IN CHARACTERIZATION THEORY

As mentioned in the Introduction of this paper, characterization theory of probability distributions on $\mathbb{R}$ or $\mathbb{R}_+$ have several results involving the convolution equation. Galambos and Kotz (1978), M.B. Rao and Shanbhag (1982), Rao (1983), Rao and Shanbhag (1986) and Davies and Shanbhag (1987) have either reviewed or implicitly discussed the literature in this connection. Klebanov (1980), Sahabov and Geshev (1974) and Alzaid *et al* (1987) are among others discussing the literature on variants or extensions of the specialized versions of the convolution equation.

There exist also results on multivariate probability distributions involving the equation (1.1) for $S = \mathbb{R}_+^n$ or $S = \mathbb{N}_0^n$. A brief account of such results is given by Davies and Shanbhag (1987). The authors of this last paper have also indicated the possibility of arriving at further characterizations of multivariate probability distributions by extending in particular the strong memoryless characterization property of the exponential distributions arrived at by Ramachandran (1979). This includes as a special case the extension of the characterization of the exponential distribution based on the memoryless property at certain specific points given by Marsaglia and Tubilla (1975) to the case of multivariate probability distributions.

The results arrived at in the present paper, besides extending the important results of Choquet and Deny (1960) and Deny (1961) to more general situations, yield several of the other characterization results falling beyond the scope of Davies and Shanbhag (1987) as special cases. In particular, the results of Section 2 of the present paper, especially Corollary 1, give the characterization discussed in Kagan *et al* (1973),

Shimizu (1969, 1978) and Davies and Shimizu (1976) of nonvanishing characteristic functions ϕ on $\mathbb{R}$ satisfying the functional equation

$$\ln\phi(t) = \int_{(0,1]} \ln\phi(tu)d\mu(u) + \int_{(0,1]} \ln\phi(-tu)d\nu(u), \qquad (5.1)$$

$$t \in \mathbb{R}_+,$$

when μ and ν are σ-finite measures satisfying some mild conditions. (To see the application of Corollary 1, express (5.1) in terms of the corresponding two equations satisfied by $-\mathrm{Re}\big(\ln\phi(t)\big)$ and $\mathrm{Im}\big(\ln\phi(t)\big)$ respectively.) The multivariate extension of the result cited is also now easy to obtain; the details about this latter characterization will be provided in a forthcoming review paper to appear in the *Festschrift* volume for P. R. Krishnaiah.

The results given in Section 3 of the present paper obviously extend the result given by Ramachandran *et al* (1987) although we have not exploited fully the line of approach of this last paper. Also these results are connected with the results of Section 2 since (3.1) is equivalent to

$$G(x) + H(x) = \int_S \big(G(x+y) + H(x+y)\big)(\mu+\nu)(dy), \quad x \in S,$$

$$G(x) - H(x) = \int_S \big(G(x+y) - H(x+y)\big)(\mu-\nu)(dy), \quad x \in S.$$

Finally, the results on a stability version of the convolution equation presented in Section 4 give as special cases, as mentioned earlier, the stability Theorem 1 of Gu and Lau (1984) and also certain versions of the stability results on the exponential and Pareto distributions given in this paper. A stability result corresponding to the multivariate generalization of Ramachandran's (1979) characterization of the exponen-

tial distributions is easy to obtain as follows. Suppose we define X and Y to be n-component random vectors as in Davies and Shanbhag (1987) on page 32 with a modification that $P\{X > Y + x | X > Y\} = P\{X > x\}$ is replaced by $P\{X > Y + x | X > Y\} = P\{X > x\}\big(1 - S(x)\big)$, where $|S(x)| \leq Ke^{-\langle \eta, x \rangle}$ with $K \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+^n$ and $\neq 0$. Then, in view of Corollary 4, after a minor manipulation, it follows that

$$P\{X > x\}\big(1 - S^*(x)\big) = \frac{1 - S^*(0)}{1 - S(0)} \int_{\mathbb{R}_+^n} \exp\{-\langle \lambda, x \rangle\} \mu(d\lambda), \quad x \in \mathbb{R}_+^n,$$

where μ is as defined in Davies and Shanbhag (1987), for some S^* such that $|S^*(x)| \leq K \exp\{-\langle \eta, x \rangle\}(1 - c)^{-1}$ with $c = E\big\{\exp\{-\langle \eta + \alpha, Y \rangle\}\big\}(P\{X > Y\})^{-1}$ whenever there exists an $\alpha \in \mathbb{R}^n$ such that $c < 1$ and $P\{X > x\}\exp\{\langle \alpha, x \rangle\}$ is component-wise decreasing on $\mathbb{R}_+^N$. One could obviously extend this result by replacing $\exp\{-\langle \eta, x \rangle\}$ by an appropriate Laplace-Stieltjes transform.

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